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Convergence of consistent and inconsistent finite difference schemes and an acceleration technique ☆

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Abstract

This paper states and generalizes in part some recent results on finite difference methods for Dirichlet problems in a bounded domain Ω which the author has obtained by himself or with coworkers. After stating a superconvergence property of finite difference solution for the case where the exact solution u belongs to $C^4(\bar{\Omega})$, it is remarked that such a property does not hold in general if $u \notin C^4(\bar{\Omega})$. Next, a convergence theorem is given for inconsistent schemes under some assumptions. Furthermore, it is shown that the accuracy of the approximate solution can be improved by a coordinate transformation. Numerical examples are also given. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Finite difference methods; Superconvergence; Nonsuperconvergence; Convergence of inconsistent scheme; Acceleration of convergence

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with the boundary Γ and consider the Dirichlet problem

$$-\Delta u + c(x, y)u = f(x, y) \quad \text{in } \Omega, \tag{1.1}$$

$$u = g(x, y) \quad \text{on } \Gamma, \tag{1.2}$$

where c, f and g are given functions and $c \geqslant 0$.

In a series of papers [3–5,12,19,20], the author, together with his coworkers, studied convergence of the Shortley–Weller (S-W) finite difference approximation applied to (1.1)–(1.2).

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The purpose of this paper is to state and generalize in part the results. First, in Section 2, we state the superconvergence result for the problem

$$\mathscr{L}u \equiv -\Delta u + \boldsymbol{b}(x,y)\cdot\nabla u + c(x,y)u = f(x,y) \quad \text{in } \Omega, \tag{1.3}$$

$$u = g(x,y) \quad \text{on } \Gamma, \tag{1.4}$$

where $\boldsymbol{b} = (b_1(x,y), b_2(x,y))$ is bounded in $\bar{\Omega}$, which includes (1.1)–(1.2) as a special case.

Next, in Section 3, we remark that the result does not hold in general if $u \notin C^4(\bar{\Omega})$. In Section 4, a convergence theorem is given for the case where the solution has singular derivatives on the boundary Γ and the truncatoin error is $O(h^{\sigma-2})$, $0 < \sigma < 2$, with the mesh size h approaching to zero. The theorem asserts that $|u(P) - U(P)| = O(h^{\sigma})$ at every grid point $P \in \Omega$, where $U(P)$ stands for the finite difference solution. Furthermore, in Section 5, we shall prove that the accuracy $O(h^{\sigma})$ can be improved by a coordinate transformation.

In Section 6, numerical examples are given to illustrate our results stated in Sections 3 and 4.

Finally, in Section 7, it is remarked that the technique employed in Section 5, which generates adaptive grid points, has actively been studied recently, especially in the area of computational fluid dynamics.

2. Superconvergence of the S-W approximation

We construct a net over $\bar{\Omega} = \Omega \cup \Gamma$ by the grid points $P_{ij} = (x_i, y_j)$ in $\bar{\Omega}$ with the equal mesh size h in the x and y directions.

The set of grid points in Ω are denoted by Ω_h. In addition, we denote by Γ_h the set of points of intersection of grid lines with Γ. Let $\hat{\Gamma}$ be a part or the whole of Γ and K a constant with $K > 1$, which is arbitrarily chosen independent of h. We define

$$\mathscr{S}_h(K,\hat{\Gamma}) = \{P \in \Omega_h \mid \operatorname{dist}(P,\hat{\Gamma}) \leqslant Kh\}.$$

If $\hat{\Gamma} = \Gamma$, then we write $\mathscr{S}_h(K)$ in place of $\mathscr{S}_h(K,\Gamma)$. We define the neighbors of $P \in \Omega_h$ to be four points in $\bar{\Omega}_h = \bar{\Omega}_h \cup \Gamma_h$ which are adjacent to P and on horizontal and vertical grid lines through P. These points are denoted by P_E, P_W, P_S, P_N and their distances to P by h_E, h_W, h_S, h_N, respectively (cf. Figs. 1 and 2). We denote by $U(P)$ the approximate solution to $u(P)$ at $P \in \Omega_h$. Then the S-W formula

$$\begin{aligned}-\Delta_h u(P) \equiv{} & \left(\frac{2}{h_E h_W} + \frac{2}{h_S h_N}\right)U(P) - \frac{2}{h_E(h_E + h_W)}U(P_E) \\ & - \frac{2}{h_W(h_E + h_W)}U(P_W) - \frac{2}{h_S(h_S + h_N)}U(P_S) \\ & - \frac{2}{h_N(h_S + h_N)}U(P_N)\end{aligned}$$

is used to approximate $-\Delta u(P)$. Furthermore, $\boldsymbol{b}(P)\cdot\nabla u(P)$ is approximated by the usual formula

$$b_1(P)\frac{u(P_E) - u(P_W)}{h_E + h_W} + b_2(P)\frac{u(P_N) - u(P_S)}{h_N + h_S}.$$

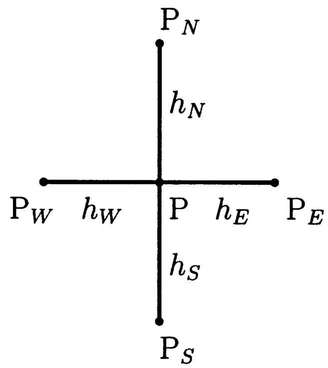

Fig. 1.

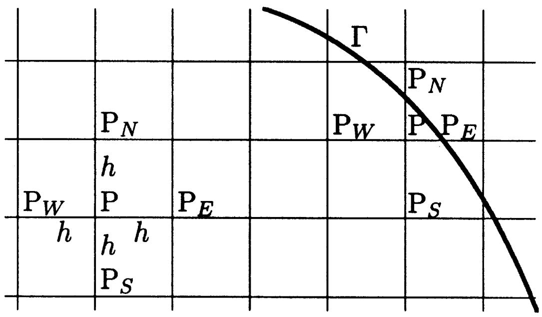

Fig. 2.

Then (1.3)–(1.4) is approximated by

$$\mathscr{L}_h U(P) = f(P), \quad P \in \Omega_h, \tag{2.1}$$

$$U(P) = g(P), \quad P \in \Gamma_h, \tag{2.2}$$

where

$$\begin{aligned}\mathscr{L}_h U(P) = {} & \left(\frac{2}{h_\mathrm{E} h_\mathrm{W}} + \frac{2}{h_\mathrm{S} h_\mathrm{N}} + c(P)\right) U(P) - \frac{1}{h_\mathrm{E}(h_\mathrm{E}+h_\mathrm{W})}\{2 - h_\mathrm{E} b_1(P)\} U(P_\mathrm{E}) \\ & - \frac{1}{h_\mathrm{W}(h_\mathrm{E}+h_\mathrm{W})}\{2 + h_\mathrm{W} b_1(P)\} U(P_\mathrm{W}) - \frac{1}{h_\mathrm{N}(h_\mathrm{S}+h_\mathrm{N})}\{2 - h_\mathrm{N} b_2(P)\} U(P_\mathrm{N}) \\ & - \frac{1}{h_\mathrm{S}(h_\mathrm{S}+h_\mathrm{N})}\{2 + h_\mathrm{S} b_2(P)\} U(P_\mathrm{S}).\end{aligned} \tag{2.3}$$

This leads to a system of linear equations

$$A\boldsymbol{U} = \tilde{\boldsymbol{f}}, \tag{2.4}$$

with respect to the unknown vector $\boldsymbol{U}=(U(P))$, $P\in\Omega_h$, where h is sufficiently small so as to satisfy

$$\sup_{P\in\Omega} h|b_i(P)|<2,\quad i=1,2,$$

so that A is an irreducibly diagonally dominant L-matrix (hence, A is an M-matrix). The vector $\tilde{\boldsymbol{f}}$ is determined by $f(P)$ and boundary condition (2.2).

If $u\in C^4(\bar{\Omega})$, then the local truncation error $\tau(P)$ for $\mathscr{L}_h$ is given by

$$\begin{aligned}\tau(P)&=\mathscr{L}_h u(P)-\mathscr{L}u(P)\\ &=(h_{\mathrm{E}}-h_{\mathrm{W}})[\tfrac{1}{2}b_1(P)u_{xx}(P)+\tfrac{1}{3}u_{xxx}(P)]\\ &\quad+(h_{\mathrm{N}}-h_{\mathrm{S}})[\tfrac{1}{2}b_2(P)u_{yy}(P)+\tfrac{1}{3}u_{yyy}(P)]\\ &\quad+\frac{1}{6}\frac{1}{h_{\mathrm{E}}+h_{\mathrm{W}}}[h_{\mathrm{E}}^3\{b_1(P)u_{xxx}(Q_{\mathrm{E}})+\tfrac{1}{2}u_{xxxx}(Q_{\mathrm{E}})\}\\ &\quad+h_{\mathrm{W}}^3\{b_1(P)u_{xxx}(Q_{\mathrm{W}})+\tfrac{1}{2}u_{xxxx}(Q_{\mathrm{W}})\}]\\ &\quad+\frac{1}{6}\frac{1}{h_{\mathrm{S}}+h_{\mathrm{N}}}[h_{\mathrm{S}}^3\{b_2(P)u_{yyy}(Q_{\mathrm{S}})+\tfrac{1}{2}u_{yyyy}(Q_{\mathrm{S}})\}\\ &\quad+h_{\mathrm{N}}^3\{b_2(P)u_{yyy}(Q_{\mathrm{N}})+\tfrac{1}{2}u_{yyyy}(Q_{\mathrm{N}})\}],\end{aligned}\tag{2.5}$$

where

$$Q_{\mathrm{E}}=(x+\theta h_{\mathrm{E}},y),\quad Q_{\mathrm{W}}=(x-\theta h_{\mathrm{W}},y),$$

$$Q_{\mathrm{N}}=(x,y+\theta h_{\mathrm{N}}),\quad Q_{\mathrm{S}}=(x,y-\theta h_{\mathrm{S}}),\ \ 0<\theta<1.$$

This is obtained in the same manner as in [13]. Hence we obtain from (2.5)

$$\tau(P)=\begin{cases}\mathrm{O}(h^2) & \text{if } h_{\mathrm{E}}=h_{\mathrm{W}}=h_{\mathrm{S}}=h_{\mathrm{N}}=h,\\ \mathrm{O}(h) & \text{otherwise.}\end{cases}$$

Let $\boldsymbol{u}=(u(P))$ and $\boldsymbol{\tau}=(\tau(P))$, $P\in\Omega_h$ and put

$$\|\tau\|_\infty=\sup_{P\in\Omega_h}|\tau(P)|,$$

$$|\boldsymbol{u}-\boldsymbol{U}|=(|u(P)-U(P)|),\quad |\boldsymbol{\tau}|=(|\tau(P)|),\ \text{etc.}$$

Then the relation $A(\boldsymbol{u}-\boldsymbol{U})=\boldsymbol{\tau}$ yields the estimate

$$|\boldsymbol{u}-\boldsymbol{U}|\leqslant A^{-1}|\boldsymbol{\tau}|\leqslant\|\boldsymbol{\tau}\|_\infty A^{-1}\boldsymbol{e}=\mathrm{O}(\|\boldsymbol{\tau}\|_\infty),$$

where $\boldsymbol{e}$ is the vector whose elements are all unity, since, as is easily seen, $A^{-1}\boldsymbol{e}$ is bounded (cf. [12]). Hence, if there is a grid point P such that $(h_{\mathrm{E}},h_{\mathrm{W}},h_{\mathrm{S}},h_{\mathrm{N}})\neq(h,h,h,h)$, then

$$u(P)-U(P)=\mathrm{O}(h)\quad\forall P\in\Omega_h.$$

This is essentially the same as the classical result due to Gerschgorin [6].

In 1962, Bramble–Hubbard [1] proved a sharper result:

Theorem 1. *Let $u \in C^4(\bar{\Omega})$ be the solution of*

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \ \text{ on } \Gamma.$$

Let

$$M_j = \sup_{P \in \Omega} \left\{ \left| \frac{\partial^j u(P)}{\partial x^i \partial y^{j-1}} \right| i = 0, 1, \ldots, j \right\}, \quad j = 3, 4 \tag{2.6}$$

and d be the diameter of the smallest circumscribed circle containing Ω. Then

$$|u(P) - U(P)| \leqslant \frac{M_4 d^2}{96} h^2 + \frac{2M_3}{3} h^3 = \mathrm{O}(h^2) \quad \forall P \in \Omega_h, \tag{2.7}$$

even if any grid point P exists such that $(h_{\mathrm{E}}, h_{\mathrm{W}}, h_{\mathrm{S}}, h_{\mathrm{N}}) \neq (h, h, h, h)$.

They used the discrete maximum principle and the discrete Green's third identity to prove Theorem 1. The proof can be rewritten in terms of M-matrices. We can find it in Refs. [7,8,13], etc. Recently, Matsunaga–Yamamoto [12] further improved (2.7) as

$$u(P) - U(P) = \mathrm{O}(h^3) \quad \text{if } P \in \mathscr{S}_h(K),$$

for (1.1)–(1.2), where $\mathscr{S}_h(K)$ is defined in Section 1. The result holds also for the case $\boldsymbol{b} \neq (0, 0)$, without any change to the proof, since the key of the proof in [12] is due to the fact that A is an irreducibly diagonally dominant M-matrix. Namely, we have

Theorem 2. *Let $u \in C^4(\bar{\Omega})$ be the solution* (1.3)–(1.4). *Then*

$$|u(P) - U(P)| = \begin{cases} \mathrm{O}(h^3) & P \in \mathscr{S}_h(K) \\ \mathrm{O}(h^2) & \textit{otherwise.} \end{cases}$$

Related results have been obtained for nonsmooth Dirichlet problem (Chen–Matsunaga–Yamamoto [3]) and for convection–diffusion problems (Fang–Yamamoto [5]).

Definition 3 (Yamamoto et al. [20]). We say that a discretized solution $\{U(P)\}$ has a superconvergence property near $\hat{\Gamma} \subseteq \Gamma$, if, for some constants $\sigma > 0$ and $K > 1$,

$$|u(P) - U(P)| = \begin{cases} \mathrm{O}(h^{\sigma+1}) & P \in \mathscr{S}_h(K, \hat{\Gamma}) \\ \mathrm{O}(h^{\sigma}) & \text{otherwise.} \end{cases}$$

Therefore, Theorem 2 means that the finite difference solution of (2.1)–(2.2) is superconvergent near Γ.

3. Nonsuperconvergence of the S-W approximation

Theorem 2 is proved under the assumption $u \in C^4(\bar{\Omega})$. Then, what situation occurs if $u \notin C^4(\bar{\Omega})$? This has been discussed in [20] for the centered five point finite difference method applied to the problem

$$-\Delta u = f \text{ in } \Omega = (0,1)\times(0,1), \quad u = g \text{ on } \Gamma = \partial\Omega, \tag{3.1}$$

where the following examples are given:

Example 4. Let $g=0$ and f be determined so that $u=\sqrt{x(1-x)}+\sqrt{y(1-y)}$ is the solution of (3.1). Then $u \in C(\bar{\Omega}) \cap C^{\infty}(\Omega)$ and $u \notin H^1(\Omega)$, whereas

$$|u(P)-U(P)| = \mathrm{O}(h^{1/2}) \quad \forall P \in \Omega_h.$$

No superconvergence occurs at any point near Γ.

Example 5. If $u=\sqrt{x}+y$ is the solution of (3.1) (f and g are determined so that u satisfies (3.1)), then

$$|u(P)-U(P)| = \begin{cases} \mathrm{O}(h^{3/2}) & \text{near } \hat{\Gamma} = \{(1,y)\,|\,0 \leqslant y \leqslant 1\} \\ \mathrm{O}(h^{1/2}) & \text{otherwise.} \end{cases}$$

That is, a superconvergence occurs near a side $\hat{\Gamma}$ of Γ.

Example 6. If $\sqrt{x}+\sqrt{y}$ is the solution of (3.1), then

$$|u(P)-U(P)| = \begin{cases} \mathrm{O}(h^{3/2}) & \text{near the corner } (1,1), \\ \mathrm{O}(h^{1/2}) & \text{otherwise.} \end{cases}$$

These examples show that the finite difference method works well even in the case where

$$\max_{P\in\Omega_h} |\tau(P)| \to \infty \quad \text{as } h \to 0.$$

In the next section, we shall give a convergence theorem for an inconsistent scheme, which includes Examples 4–6 as special cases.

4. Convergence of inconsistent scheme

Let $\Omega=(0,1)\times(0,1)$ and consider the problem (1.1)–(1.2), whose solution u belongs to $C(\bar{\Omega})\cap C^4(\Omega)$ and has singular derivatives near the boundary Γ such that

$$\sup_{x\in(0,1)} \frac{x^j(1-x)^j|\partial^j u/\partial x^j(x,y)|}{x^{\alpha}(1-x)^{\beta}} \leqslant K_1 < \infty \tag{4.1}$$

and

$$\sup_{y\in(0,1)} \frac{y^j(1-y)^j|\partial^j u/\partial y^j(x,y)|}{y^\gamma(1-y)^\delta} \leqslant K_2 < \infty, \quad j=2,3,4 \tag{4.2}$$

with constants $\alpha,\beta,\gamma,\delta\in(0,2)$ and $K_1,K_2>0$, where K_1 and K_2 are constants independent of y and x, respectively. To solve (1.1)–(1.2) by the centered five point formula, we put

$$h=\frac{1}{n+1},$$

$$x_i=ih, \quad i=0,1,2,\ldots,n+1$$

and

$$y_j=jh, \quad j=0,1,2,\ldots,n+1.$$

Then, it is easy to see that

$$|\tau(P)| = \begin{cases} \mathrm{O}(h^{\min(\alpha,\beta,\gamma)-2}) & (\text{near } \Gamma_1=\{(x,0)\,|\,0\leqslant x\leqslant 1\}) \\ \mathrm{O}(h^{\min(\beta,\gamma,\delta)-2}) & (\text{near } \Gamma_2=\{(1,y)\,|\,0\leqslant y\leqslant 1\}) \\ \mathrm{O}(h^{\min(\alpha,\beta,\delta)-2}) & (\text{near } \Gamma_3=\{(x,1)\,|\,0\leqslant x\leqslant 1\}) \\ \mathrm{O}(h^{\min(\alpha,\gamma,\delta)-2}) & (\text{near } \Gamma_4=\{(0,y)\,|\,0\leqslant y\leqslant 1\}) \end{cases}$$
$$\to\infty$$

as $h\to 0$. Hence the usual convergence theorem cannot be applied in such a case. However, we can prove the convergence of this scheme, provided that

$$\omega(r) \equiv \sup_{\mathrm{dist}(P,Q)\leqslant r} |u(P)-u(Q)| \leqslant C_0 r^\sigma \quad \text{at } P,Q \text{ near } \Gamma, \tag{4.3}$$

where C_0 is a positive constant and $\sigma=\min(\alpha,\beta,\gamma,\delta)$.

Theorem 7. *Under the conditions* (4.1)–(4.3), *the finite difference method applied to* (1.1)–(1.2) *converges*:

$$|u(P)-U(P)| \leqslant \mathrm{O}(h^\sigma) \quad \forall P\in\Omega_h.$$

Proof. A proof for the case $\alpha=\beta=\gamma=\delta$ is given in [4]. The same proof works for Theorem 7. But, we restate it here for the sake of the reader's convenience, since a proof of Theorem 9, which will be established in the next section, is based upon it. Let κ be a constant with $0<\kappa<\frac{1}{4}$ (say) and $I=[\kappa/h]-1$, where $[a]$ denotes the integral part of a positive number a. We put

$$\Omega_h^{(i)}=\{P\in\Omega_h\,|\,\mathrm{dist}(P,\Gamma)=ih\}, \quad i=1,2,\ldots,I$$

and

$$\Omega_h^{(0)}=\Omega_h \Big\backslash \bigcup_{i=1}^{I} \Omega_h^{(i)}.$$

The number of the element of $\Omega_h^{(i)}$ is denoted by m_i $(i=0,1,2,\dots,I)$ so that $m_0=n^2-\sum_{i=1}^{I} m_i$. We arrange the grid points of Ω_h in the order of $\Omega_h^{(1)},\Omega_h^{(2)},\dots,\Omega_h^{(I)},\Omega_h^{(0)}$. (The elements of $\Omega_h^{(i)}$ may be arranged appropriately for each $i\geqslant 0$). Let $\boldsymbol{e}^{(i)}$, $i=0,1,2,\dots,I$ be the m_i-dimensional column vectors whose elements are all unity and put $\boldsymbol{e}=(\boldsymbol{e}^{(1)t},\dots,\boldsymbol{e}^{(I)^t},\boldsymbol{e}^{(0)^t})^t$. Then a simple computation yields

$$h^2A\begin{pmatrix}\boldsymbol{e}^{(1)}\\ \boldsymbol{e}^{(2)}\\ \vdots\\ \boldsymbol{e}^{(I)}\\ \boldsymbol{e}^{(0)}\end{pmatrix}\geqslant\begin{pmatrix}\boldsymbol{e}^{(1)}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}.$$

Hence,

$$A^{-1}\begin{pmatrix}\boldsymbol{e}^{(1)}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\leqslant h^2\begin{pmatrix}\boldsymbol{e}^{(1)}\\ \boldsymbol{e}^{(2)}\\ \vdots\\ \boldsymbol{e}^{(I)}\\ \boldsymbol{e}^{(0)}\end{pmatrix}=h^2\boldsymbol{e}.$$

Furthermore, we have

$$h^2A\begin{pmatrix}\mathbf{0}\\ \boldsymbol{e}^{(2)}\\ \boldsymbol{e}^{(3)}\\ \vdots\\ \boldsymbol{e}^{(I)}\\ \boldsymbol{e}^{(0)}\end{pmatrix}\geqslant\begin{pmatrix}-\boldsymbol{e}^{(1)}\\ \boldsymbol{e}^{(2)}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}$$

so that

$$A^{-1}\begin{pmatrix}\mathbf{0}\\ \boldsymbol{e}^{(2)}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\leqslant h^2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{e}^{(2)}\\ \boldsymbol{e}^{(3)}\\ \vdots\\ \boldsymbol{e}^{(I)}\\ \boldsymbol{e}^{(0)}\end{pmatrix}+A^{-1}\begin{pmatrix}\boldsymbol{e}^{(1)}\\ \mathbf{0}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\leqslant h^2\begin{pmatrix}\boldsymbol{e}^{(1)}\\ 2\boldsymbol{e}^{(2)}\\ 2\boldsymbol{e}^{(3)}\\ \vdots\\ 2\boldsymbol{e}^{(I)}\\ 2\boldsymbol{e}^{(0)}\end{pmatrix}\leqslant 2h^2\boldsymbol{e}.$$

Repeating this argument, we obtain

$$A^{-1}\begin{pmatrix}\mathbf{0}\\ \vdots\\ \mathbf{0}\\ \boldsymbol{e}^{(i)}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{pmatrix} \leqslant h^2\begin{pmatrix}(i-1)\boldsymbol{e}^{(1)}\\ \vdots\\ (i-1)\boldsymbol{e}^{(i-1)}\\ i\boldsymbol{e}^{(i)}\\ i\boldsymbol{e}^{(i+1)}\\ \vdots\\ i\boldsymbol{e}^{(0)}\end{pmatrix} \leqslant ih^2\boldsymbol{e}.$$

On the other hand, it follows from assumptions (4.1)–(4.3) that with a positive constant C which is independent of i and h,

$$\|\boldsymbol{\tau}^{(i)}\|_\infty = \sup_{P\in\Omega_h^{(i)}} |\tau(P)| \leqslant C\cdot(ih)^{\sigma-4}\cdot h^2 = \frac{C}{i^{4-\sigma}}h^{\sigma-2}.$$

In fact, if $P\in\Omega_h^{(i)}$, $i\geqslant 2$, then $|\tau(P)| = \mathrm{O}(((i-1)h)^{\sigma-4}h^2) = \mathrm{O}((ih)^{\sigma-4}h^2)$. For the case $i=1$, we also see from (4.1)–(4.3) that if $P\in\Omega_h^{(1)}$,

$$\begin{aligned}|\tau(P)| &= |-\Delta_h u(P) + \Delta u(P)|\\ &\leqslant |\Delta_h u(P)| + |\Delta u(P)|\\ &= \mathrm{O}(h^{\sigma-2}) + \mathrm{O}(h^{\sigma-2}) \leqslant C'h^{\sigma-2},\end{aligned}$$

where C' is a positive constant. Hence

$$\begin{aligned}|\boldsymbol{u}-\boldsymbol{U}| &\leqslant A^{-1}\begin{pmatrix}\|\boldsymbol{\tau}^{(1)}\|_\infty\boldsymbol{e}^{(1)}\\ \vdots\\ \|\boldsymbol{\tau}^{(I)}\|_\infty\boldsymbol{e}^{(I)}\\ \|\boldsymbol{\tau}^{(0)}\|_\infty\boldsymbol{e}^{(0)}\end{pmatrix}\\ &= \sum_{i=1}^{I}\|\boldsymbol{\tau}^{(i)}\|_\infty A^{-1}\begin{pmatrix}\mathbf{0}\\ \vdots\\ \boldsymbol{e}^{(i)}\\ \vdots\\ \mathbf{0}\end{pmatrix} + \|\boldsymbol{\tau}^{(0)}\|_\infty A^{-1}\begin{pmatrix}\mathbf{0}\\ \vdots\\ \vdots\\ \mathbf{0}\\ \boldsymbol{e}^{(0)}\end{pmatrix}\\ &\leqslant C''\left(\sum_{i=1}^{I}\frac{h^{\sigma-2}}{i^{4-\sigma}}i\right)h^2\boldsymbol{e} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e} \quad (C'' = \max(C, C'))\\ &= C''\left(\sum_{i=1}^{I}\frac{1}{i^{3-\sigma}}\right)h^\sigma\boldsymbol{e} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e}.\end{aligned}$$

Since $3-\sigma>1$, we have

$$\sum_{i=1}^{I}\frac{1}{i^{3-\sigma}}<\sum_{i=1}^{\infty}\frac{1}{i^{3-\sigma}}<\infty.$$

Furthermore, as was remarked in Section 2, $A^{-1}\boldsymbol{e}$ is bounded. Consequently, we have

$$|u(P)-U(P)|\leqslant \mathrm{O}(h^{\sigma})\quad \forall P\in\Omega_h. \qquad \square$$

Remark 8. During the author's visit to Charles University, Prague in November 14–21, 2000, the author learned that Prof. M. Feisfauer had also obtained a result similar to Theorem 7 with the use of the maximum principle. Roughly speaking, his result is stated as follows: If $\tau(P)=\mathrm{O}(h^{\sigma-2})$ at every grid point P adjacent to Γ and $0<\sigma<2$, then $u(P)-U(P)=\mathrm{O}(h^{\sigma})$, $\forall P\in\Omega_h$.

5. Acceleration of convergence

In this section, we shall show that we can improve the accuracy $\mathrm{O}(h^{\sigma})$ in Theorem 7 by a coordinate transformation, under conditions (4.1)–(4.3). Let $\varphi(t)$ be the function defined by

$$\varphi(0)=0,\quad \varphi(1)=1,$$

$$\varphi'(t)=c_p\{t(1-t)\}^p,$$

where $p\geqslant 0$, that is,

$$\varphi(t)=c_p\int_0^t\{s(1-s)\}^p\,\mathrm{d}s,\quad c_p=\left[\int_0^1\{s(1-s)\}^p\,\mathrm{d}s\right]^{-1}. \tag{5.1}$$

Observe that $\varphi(t)=t$ if $p=0$. We then put

$$h=\frac{1}{n+1},\qquad t_i=ih,\quad i=0,1,2,\ldots,n+1$$

$$x_i=\varphi(t_i),\qquad y_j=\varphi(t_j),\quad j=0,1,2,\ldots,n+1$$

and generate nonuniform grid points $P_{ij}=(x_i,y_j)$. The set of the grid points are again denoted by Ω_h. The set Γ_h is also defined as before. Then, at $P=P_{ij}$, $1\leqslant i,\ j\leqslant n$, we obtain from (2.5)

$$h_{\mathrm{E}}=x_{i+1}-x_i=h\varphi'(t_i+\theta_i^+h)=\mathrm{O}(i^ph^{p+1})<h\varphi'(t_{i+1}),\quad 0<\theta_i^+<1, \tag{5.2}$$

$$h_{\mathrm{W}}=x_i-x_{i-1}=h\varphi'(t_i-\theta_i^-h)=\mathrm{O}(i^ph^{p+1})<h\varphi'(t_i),\quad 0<\theta_i^-<1, \tag{5.3}$$

$$\begin{aligned}h_{\mathrm{E}}-h_{\mathrm{W}}&=x_{i+1}-2x_i+x_{i-1}\\&=h^2\varphi''(t_i+\theta_i^1h)=\mathrm{O}(i^{p-1}h^{p+1}),\quad -1<\theta_i^1<1,\end{aligned} \tag{5.4}$$

$$\begin{aligned}h_{\mathrm{N}}-h_{\mathrm{S}}&=y_{j+1}-2y_j+y_{j-1}\\&=h^2\varphi''(t_j+\theta_j^2h)=\mathrm{O}(j^{p-1}h^{p+1}),\quad -1<\theta_j^2<1,\ \text{etc.}\end{aligned} \tag{5.5}$$

An attention may be necessary for the case $i=1$ in (5.4) or $j=1$ in (5.5). For example, to prove (5.4) for the case $i=1$, it suffices to observe that

$$\lim_{h\to 0}\frac{1}{h^{p+1}}(x_2-2x_1+x_0)=\lim_{h\to 0}\frac{c_p}{h^{p+1}}\left[\int_0^{2h}\{s(1-s)\}^p\,\mathrm{d}s-2\int_0^{h}\{s(1-s)\}^p\,\mathrm{d}s\right]$$
$$=\frac{c_p}{p+1}(2^{p+1}-2)<\infty.$$

We are now in a position to prove the following result:

Theorem 9. *Under conditions* (4.1)–(4.3), *apply the S-W approximation to the problem* (1.1)–(1.2). *Then, if* $r=\sigma(p+1)<2$, *then*

$$|u(P)-U(P)|\leqslant \mathrm{O}(h^r)\quad \forall P\in\Omega_h. \tag{5.6}$$

If $r=2$, *then the estimate* (5.6) *is replaced by*

$$|u(P)-U(P)|\leqslant \mathrm{O}\left(h^2\log\frac{1}{h}\right)\quad \forall P\in\Omega_h. \tag{5.7}$$

Proof. The case $p=0$ reduces to Theorem 7. Hence, we put $p>0$. We again take a positive number $\kappa<\frac{1}{4}$ and set $I=[\kappa/h]$. Furthermore, we define $\Omega_h^{(i)}$ inductively by

$$\Omega_h^{(1)}=\{P\in\Omega_h\,|\text{ at least one of the neighbors of } P \text{ belongs to } \Gamma_h\},$$

$$\Omega_h^{(i)}=\left\{P\in\Omega_h^{(i)}\Big\backslash\bigcup_{j=1}^{i-1}\Omega_h^{(j)}\;\middle|\;\text{at least one of the neighbors of } P\text{ belongs to } \Omega_h^{(i-1)}\right\}\quad (i\geqslant 2)$$

and put

$$\Omega_h^{(0)}=\Omega_h\Big\backslash\bigcup_{i=1}^{I}\Omega_h^{(i)}.$$

The vectors $\boldsymbol{e}^{(i)}$, $i=0,1,\ldots,I$ and $\boldsymbol{e}$ are also defined as before. Then, the element of $A\boldsymbol{e}$ corresponding to $P=(x,y)\in\Omega_h$ is given by

$$(A\boldsymbol{e})_P=0\quad\text{if } P\notin\Omega_h^{(1)}$$

and

$$(A\boldsymbol{e})_P\geqslant\begin{cases}\frac{2}{h_\mathrm{E}h_\mathrm{W}}-\frac{2}{h_\mathrm{E}(h_\mathrm{E}+h_\mathrm{W})}=\frac{2}{h_\mathrm{W}(h_\mathrm{E}+h_\mathrm{W})} & (P\in\Omega_h^{(1)},\ x=x_1),\\ \frac{2}{h_\mathrm{E}h_\mathrm{W}}-\frac{2}{h_\mathrm{W}(h_\mathrm{E}+h_\mathrm{W})}=\frac{2}{h_\mathrm{E}(h_\mathrm{E}+h_\mathrm{W})} & (P\in\Omega_h^{(1)},\ x=x_n),\\ \frac{2}{h_\mathrm{S}h_\mathrm{N}}-\frac{2}{h_\mathrm{N}(h_\mathrm{S}+h_\mathrm{N})}=\frac{2}{h_\mathrm{S}(h_\mathrm{S}+h_\mathrm{N})} & (P\in\Omega_h^{(1)},\ y=y_1),\\ \frac{2}{h_\mathrm{S}h_\mathrm{N}}-\frac{2}{h_\mathrm{S}(h_\mathrm{S}+h_\mathrm{N})}=\frac{2}{h_\mathrm{N}(h_\mathrm{S}+h_\mathrm{N})} & (P\in\Omega_h^{(1)},\ y=y_n).\end{cases}$$

Hence, putting $h_{\mathrm{E}}^i = x_{i+1} - x_i$, $h_{\mathrm{W}}^i = x_i - x_{i-1}$ and $d_i = 2/(h_{\mathrm{E}}^i + h_{\mathrm{W}}^i)$, $i = 1, 2, \ldots, I$, we obtain

$$A\boldsymbol{e} \geqslant \frac{d_1}{h_{\mathrm{W}}^1}\begin{pmatrix} \boldsymbol{e}^{(1)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \quad \text{or} \quad A^{-1}\begin{pmatrix} d_1\boldsymbol{e}^{(1)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \leqslant h_{\mathrm{W}}^1\boldsymbol{e},$$

Similarly, we have for $i \geqslant 2$

$$A\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}^{(i)} \\ \boldsymbol{e}^{(i+1)} \\ \vdots \\ \boldsymbol{e}^{(0)} \end{pmatrix} \geqslant \begin{pmatrix} \mathbf{0} \\ \vdots \\ -d_{i-1}/h_{\mathrm{E}}^{i-1}\boldsymbol{e}^{(i-1)} \\ d_i/h_{\mathrm{W}}^i\boldsymbol{e}^{(i)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}.$$

Since $h_{\mathrm{E}}^{i-1} = h_{\mathrm{W}}^i$, we obtain

$$A^{-1}\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ d_i\boldsymbol{e}^{(i)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \leqslant h_{\mathrm{W}}^i\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}^{(i)} \\ \boldsymbol{e}^{(i+1)} \\ \vdots \\ \boldsymbol{e}^{(0)} \end{pmatrix} + A^{-1}\begin{pmatrix} \mathbf{0} \\ \vdots \\ d_{i-1}\boldsymbol{e}^{(i-1)} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

$$\leqslant h_{\mathrm{W}}^i\begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}^{(i)} \\ \boldsymbol{e}^{(i+1)} \\ \vdots \\ \boldsymbol{e}^{(0)} \end{pmatrix} + h_{\mathrm{W}}^{i-1}\begin{pmatrix} \mathbf{0} \\ \vdots \\ \boldsymbol{e}^{(i-1)} \\ \boldsymbol{e}^{(i)} \\ \vdots \\ \vdots \\ \boldsymbol{e}^{(0)} \end{pmatrix} + \cdots + h_{\mathrm{W}}^2\begin{pmatrix} \mathbf{0} \\ \boldsymbol{e}^{(2)} \\ \boldsymbol{e}^{(3)} \\ \vdots \\ \vdots \\ \vdots \\ \boldsymbol{e}^{(0)} \end{pmatrix} + A^{-1}\begin{pmatrix} d_1\boldsymbol{e}^{(1)} \\ \mathbf{0} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{0} \end{pmatrix}$$

$$\leqslant h_{\mathrm{W}}^i\boldsymbol{e} + h_{\mathrm{W}}^{i-1}\boldsymbol{e} + \cdots + h_{\mathrm{W}}^2\boldsymbol{e} + h_{\mathrm{W}}^1\boldsymbol{e} = \varphi(t_i)\boldsymbol{e}.$$

Furthermore, if $i \geqslant 2$, then it follows from (4.1)–(4.3) and (5.2)–(5.5) that at $P = P_{kl} = (\varphi(t_k), \varphi(t_l)) \in \Omega_h^{(i)}$, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
|\tau(P)| &\leqslant \tfrac{1}{3}|h_\mathrm{E} - h_\mathrm{W}|\,|u_{xxx}(P)| + \tfrac{1}{3}|h_\mathrm{N} - h_\mathrm{S}|\,|u_{yyy}(P)| \\
&\quad + \frac{1}{12}\frac{1}{h_\mathrm{E} + h_\mathrm{W}}\{h_\mathrm{E}^3|u_{xxxx}(Q_\mathrm{E})| + h_\mathrm{W}^3|u_{xxxx}(Q_\mathrm{W})|\} \\
&\quad + \frac{1}{12}\frac{1}{h_\mathrm{S} + h_\mathrm{N}}\{h_\mathrm{S}^3|u_{yyyy}(Q_\mathrm{S})| + h_\mathrm{N}^3|u_{yyyy}(Q_\mathrm{N})|\} \\
&\leqslant C_1[(h^2(ih)^{p-1}) \cdot ((ih)^{p+1})^{\sigma-3} + (h(ih)^p)^2 \cdot ((ih)^{p+1})^{\sigma-4}] \\
&= 2C_1 i^{(\sigma-2)(p+1)-2} h^{(p+1)(\sigma-2)} \\
&\equiv \tau_\infty^{(i)} \quad \text{(say)},
\end{aligned} \tag{5.8}$$

where we have used the fact that at $P = P_{kl} \in \Omega_h^{(i)}$

$$0 < h_\mathrm{E} - h_\mathrm{W} < h^2\varphi''(t_{i+1}) = h^2(\mathrm{O}(ih)^{p-1}), \quad |u_{xxx}(P)| \leqslant \mathrm{O}(x_i^{\sigma-3}), \text{ etc.}$$

Hence,

$$\|\boldsymbol{\tau}^{(i)}\|_\infty = \sup_{P \in \Omega_h^{(i)}} |\tau(P)| \leqslant \tau_\infty^{(i)}, \quad 2 \leqslant i \leqslant I. \tag{5.9}$$

If $i = 1$ and $P \in \Omega_h^{(1)}$, then we obtain from (4.1)–(4.3)

$$\begin{aligned}
\tau(P) = -\Delta_h u(P) + \Delta u(P) &= \frac{2h_\mathrm{W}\{u(P) - u(P_\mathrm{E})\} + 2h_\mathrm{E}\{u(P) - u(P_\mathrm{W})\}}{h_\mathrm{E}h_\mathrm{W}(h_\mathrm{E} + h_\mathrm{W})} \\
&\quad + \frac{2h_\mathrm{S}\{u(P) - u(P_\mathrm{N})\} + 2h_\mathrm{N}\{u(P) - u(P_\mathrm{S})\}}{h_\mathrm{S}h_\mathrm{N}(h_\mathrm{S} + h_\mathrm{N})} + \Delta u(P)
\end{aligned}$$

and

$$\begin{aligned}
|\tau(P)| &\leqslant \frac{2C_0(h_\mathrm{E}^\sigma + h_\mathrm{W}^\sigma)}{h_\mathrm{E}h_\mathrm{W}} + \frac{2C_0(h_\mathrm{S}^\sigma + h_\mathrm{N}^\sigma)}{h_\mathrm{S}h_\mathrm{N}} + \mathrm{O}(\mathrm{dist}(P, \Gamma)^{\sigma-2}) \\
&\leqslant \mathrm{O}((h^{p+1})^{\sigma-2}) + \mathrm{O}((h^{p+1})^{\sigma-2}) + \mathrm{O}((h^{p+1})^{\sigma-2}) \\
&= \mathrm{O}(h^{(p+1)(\sigma-2)}).
\end{aligned}$$

Therefore, (5.8) as well as (5.9) hold for $i = 1$ by choosing a larger constant C_1 if necessary.

Consequently, as in the proof of Theorem 7,

$$|\boldsymbol{u}-\boldsymbol{U}| \leqslant \sum_{i=1}^{I} \|\boldsymbol{\tau}^{(i)}\|_\infty A^{-1} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}^{(i)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \|\boldsymbol{\tau}^{(0)}\|_\infty A^{-1} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}^{(0)} \end{pmatrix}$$

$$\leqslant \sum_{i=1}^{I} \tau_\infty^{(i)} \frac{h_{\mathrm{E}}^i + h_{\mathrm{W}}^i}{2} A^{-1} \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ d_i\boldsymbol{e}^{(i)} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e}$$

$$\leqslant \sum_{i=1}^{I} \tau_\infty^{(i)} \frac{\varphi(t_{i+1}) - \varphi(t_{i-2})}{2} \varphi(t_i)\boldsymbol{e} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e}$$

$$\leqslant \sum_{i=1}^{I} \tau_\infty^{(i)} [h\varphi'(t_{i+1})]\varphi(t_i)\boldsymbol{e} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e}. \tag{5.10}$$

Furthermore, we obtain from (5.8)

$$\begin{aligned} \tau_\infty^{(i)}[h\varphi'(t_{i+1})]\varphi(t_i) &\leqslant 2C_1 i^{(\sigma-2)(p+1)-2} h^{(p+1)(\sigma-2)} [c_p h\{(i+1)h\}^p] c_p \frac{(ih)^{p+1}}{p+1} \\ &\leqslant 2^{p+1} C_1 c_p^2 i^{(\sigma-2)(p+1)-1+2p} h^{(p+1)\sigma} \\ &= C_2 i^q h^{(p+1)\sigma}, \quad 1 \leqslant i \leqslant I \end{aligned} \tag{5.11}$$

with $C_2 = 2^{p+1}C_1c_p^2$ and $q = \sigma(p+1) - 3$, where we have used a rough estimate $(i+1)^p \leqslant 2^p i^p$. Then

$$|\boldsymbol{u}-\boldsymbol{U}| \leqslant C_2 \left(\sum_{i=1}^{I} i^q \right) h^{(p+1)\sigma}\boldsymbol{e} + \mathrm{O}(h^2)A^{-1}\boldsymbol{e}$$

and

$$\sum_{i=1}^{I} i^q \leqslant \sum_{i=1}^{\infty} i^q < \infty$$

if $q < -1$, or equivalently, if $\sigma(p+1) < 2$. Next, to prove the boundedness of the vector $A^{-1}\boldsymbol{e}$, it suffices to repeat the same argument as in the proof of Theorem 2.2 in [12]. That is, we consider a domain D such that $D \supset \Omega$ and the boundary ∂D is sufficiently smooth so that the Dirichlet problem

$$\begin{aligned} -\Delta v &= 1 \quad \text{in } D, \\ v &= 0 \quad \text{on } \partial D \end{aligned}$$

has a (unique) solution $v \in C^{2,\lambda}(\bar{D})$, where λ is the Hölder index with $0 < \lambda < 1$. Then, we can prove that

$$A^{-1}\boldsymbol{e} \leqslant 2\boldsymbol{v},$$

where $\boldsymbol{v}$ is the n^2-dimensional vector whose components are $v(P)$, $P \in \Omega_h$.

We thus conclude that, under the conditions $q < -1$,

$$\begin{aligned} |u(P) - U(P)| &\leqslant \mathrm{O}(h^{\sigma(p+1)}) + \mathrm{O}(h^2) \\ &= \mathrm{O}(h^r), \end{aligned} \tag{5.12}$$

where

$$r = \sigma(p+1).$$

Furthermore, if $q = -1$ or equivalently, $r = 2$, then

$$\sum_{i=1}^{I} i^q \leqslant 1 + \int_1^I \frac{1}{x}\,\mathrm{d}x = \mathrm{O}(\log I) = \mathrm{O}\left(\log \frac{1}{h}\right).$$

This completes the proof of Theorem 9. □

Remark 10. It is possible to extend Theorems 7 and 9 to a more general domain. This will be discussed elsewhere.

6. Numerical example

We employ the finite difference method with uniform grids and the transformation φ for acceleration to solve the problem

$$\begin{aligned} -\Delta u &= f(x,y) \quad \text{in } \Omega = (0,1) \times (0,1), \\ u &= 0 \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

where f is determined so that $u(x,y) = \sqrt{xy}(1-x)(1-y)$ is the solution. In this case, we have $\alpha = \gamma = \frac{1}{2}$, $\beta = \delta = 1$ and $\sigma = \frac{1}{2}$. For the sake of comparison, we tested several p's. The results of computation done by Y. Shogenji of Ehime University are shown in Table 1. We see from Table 1 that the finite difference solution applied to the problem and accelerated ones converge to the solution with $\mathrm{O}(h^{1/2})$ and $\mathrm{O}(h^r)$ (or $\mathrm{O}(h^2 \log 1/h)$) accuracy indicated in Theorems 7 and 9, respectively.

Table 1
Effect of transformation ($\varepsilon = \max_{P \in \Omega_h} |u(P) - U(P)|$)

p	$r=(p+1)/2$	h	ε	ε/h^r
0	0.5	1/100	1.33589×10^{-2}	1.33589×10^{-1}
		1/200	9.59659×10^{-3}	1.35716×10^{-1}
		1/300	7.89750×10^{-3}	1.36789×10^{-1}
0.2	0.6	1/100	9.13296×10^{-3}	1.44748×10^{-1}
		1/200	6.13171×10^{-3}	1.47299×10^{-1}
		1/300	4.84173×10^{-3}	1.48345×10^{-1}
0.4	0.7	1/100	6.16989×10^{-3}	1.54981×10^{-1}
		1/200	3.86844×10^{-3}	1.57854×10^{-1}
		1/300	2.93376×10^{-3}	1.59004×10^{-1}
0.5	0.75	1/100	5.03741×10^{-3}	1.59297×10^{-1}
		1/200	3.05120×10^{-3}	1.62272×10^{-1}
		1/300	2.26778×10^{-3}	1.63471×10^{-1}
0.6	0.8	1/100	4.08534×10^{-3}	1.62640×10^{-1}
		1/200	2.39124×10^{-3}	1.65748×10^{-1}
		1/300	1.74190×10^{-3}	1.67002×10^{-1}
0.8	0.9	1/100	2.61713×10^{-3}	1.65130×10^{-1}
		1/200	1.42938×10^{-3}	1.68296×10^{-1}
		1/300	1.00000×10^{-3}	1.69594×10^{-1}
1	1	1/100	1.58394×10^{-3}	1.58394×10^{-1}
		1/200	8.06056×10^{-4}	1.61211×10^{-1}
		1/300	5.41301×10^{-4}	1.62390×10^{-1}
2	1.5	1/100	4.07420×10^{-4}	4.07420×10^{-1}
		1/200	1.62517×10^{-4}	4.59667×10^{-1}
		1/300	9.31762×10^{-5}	4.84157×10^{-1}
3	2	1/100	6.34004×10^{-4}	6.34004×10^{0}
		1/200	1.96881×10^{-4}	7.87523×10^{0}
		1/300	9.75450×10^{-5}	8.77905×10^{0}

7. Concluding remarks

The superconvergence property described in Sections 2 and 3 is an interesting and remarkable property of FDM, which suggests the possibility of constructing an algorithm with $O(h^2)$-accuracy for Dirichlet–Neumann map in a general domain. Although superconvergence property is also known in FEM (e.g., [18]), the property in FEM is slightly different from that of FDM. In fact, FEM theory does not necessarily indicate the increase of the accuracy of FEM solution near the boundary. This is probably due to the fact that the corresponding stiffness matrix is not an M-matrix in general. Furthermore, Theorem 7 suggests that FDM might work well for the solution $u \notin H^1(\Omega)$, while such a function is out of the framework of FEM theory.

A method which uses appropriate coordinate transformations to generate nonuniform grids has been known in the area of computational fluid dynamics and is a basis of recent methods for adaptive grid generation for steady and unsteady problems (cf. Refs. [17,9,2], etc.). However, it appears that little mention has been made concerning the error estimation of the numerical solutions except for one-dimensional case (cf. [15,16]). Hence, results of this paper provide a new insight into the accuracy of numerical solutions obtained by the usual finite difference method and a coordinate transformation method in the case where the solution has singular derivatives near the boundary. Of course, we do not know a priori the exact solution of the problem so that we cannot determine the optimal value of the parameter p in practical computation. Therefore, our results may be qualitative. It should be emphasized, however, that our results give a mathematical justification for adaptive grid methods for the case where Ω is a square or a more general domain.

Finally, we note that as practical and useful methods in domains with irregular boundaries, second order accurate methods, called the immersed boundary methods, immersed interface methods, etc., are proposed and developed by Peskin [14], Leveque-Li [10,11], etc., which are successfully applied to some problems with discontinuous coefficients or singular source terms so that the solutions are not smooth near boundary or across some interface.

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